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LETTER TO THE EDITOR

The relation between amplitudes and critical exponents in finite-size scaling

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Abstract. The relation between critical exponents and the amplitude of the correlation length divergence at the critical point of two-dimensional systems as a function of finite size is investigated and generalised in two ways. Correlations more general than those of order–order type are included. A form appropriate for anisotropic systems is proposed. We present (a) exact results for the Ising and Gaussian models and (b) numerical results for the symmetric eight-vertex (Baxter), continuous q-state Potts, and continuous N-component cubic models.

In this letter we consider two-dimensional lattices, infinite in one direction and of finite size n in the other. Denote by κ_n the inverse correlation length, in the sense defined below, of the system in the infinite direction. Suppose that all parameters such as temperature and symmetry-breaking field are set to the critical values of the truly infinite system. If we furthermore assume the case of a continuous transition, then κ_n behaves for large n as

$$\kappa_n \simeq A/n. \tag{1}$$

The divergence of the correlation length as shown by this equation is a fundamental feature of a phase transition. In numerical studies in critical phenomena, accurate estimates of critical properties can be derived from the behaviour of the correlation length as a function of system size with the help of finite-size scaling or phenomenological renormalisation (Nightingale 1982 and references therein).

In a study of the two-dimensional XY model Luck (1982) derived the remarkable relation‖

$$A = 2\pi x, \tag{2}$$

where x, the anomalous dimension, is the exponent which describes how the spin–spin correlation function g of the infinite system at criticality decays as a function of distance r:

$$g(r) \sim r^{-2x}. \tag{3}$$

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‖ This relation was first demonstrated by J L Pichard and G Farma (1981 *J. Phys. C: Solid State Phys.* **14** L617) in the case of Anderson localisation.

Using (2), one need only calculate A to find x; no derivatives of κ_n are required as in the more usual finite-size scaling procedure. Equation (2) has been numerically verified for the spin–spin correlation function of the q-state Potts model in two dimensions (Derrida and de Seze 1982). Results were published only for two special cases: percolation and the Ising model, i.e. $q=1$ and $q=2$.

Luck's result, equation (2), can be reproduced easily (Thouless 1982) with the spin-wave approximation to the two-dimensional XY model, i.e. the Gaussian model As it stands (2) applies to isotropic models only. To investigate the effect of anisotropy we explicitly consider the anisotropic Gaussian model. The reduced Hamiltonian (i.e. with a factor $-1/k_{\mathrm{B}}T$ included) for this model is

$$\mathscr{H}=-K_1\sum\nolimits^{(1)}[\phi(\boldsymbol{r})-\phi(\boldsymbol{r}')]^2-K_2\sum\nolimits^{(2)}[\phi(\boldsymbol{r})-\phi(\boldsymbol{r}')]^2 \tag{4}$$

where the first sum is over all nearest-neighbour bonds in one direction and the second sum is the same in the other direction of a square lattice. The continuous variables ϕ assume values from $-\infty$ to ∞. The analogue of the spin–spin correlation function in the XY model for a strip of width n on a cylinder is given by

$$\langle\cos\phi(\mathbf{0})\cos\phi(\boldsymbol{r})\rangle\sim r^{-2x}\,\mathrm{e}^{-A_1 r/n}. \tag{5}$$

Here we assume that the strip is infinite in the direction of K_1 and also that $\boldsymbol{r}$ points along this direction. The exponent and amplitude in (5) are

$$x=1/4\pi(K_1K_2)^{1/2} \qquad \text{and} \qquad A_1=1/2K_1. \tag{6}$$

Equations (5) and (6) follow from (20) of Cardy and Nightingale (1983), the generalisation of which to anisotropic interactions is straightforward. Since (5) and (6) also hold with K_1 and K_2 interchanged, one finds

$$A\equiv(A_1A_2)^{1/2}=2\pi x, \tag{7}$$

where A_2 is the amplitude of the inverse correlation length when K_2 is in the infinite direction. This is the generalisation to anisotropic lattices of equation (2).

For the case that the strip makes an arbitrary angle θ with the direction of K_1, we may make use of known properties of the Gaussian model in continuous space. After a transformation to a coordinate system with the y axis parallel to the infinite direction, and a Fourier transformation, a cross term proportional to the wavenumbers k_x and k_y appears. Since $k_x=0$ is the dominant term, the cross term vanishes and it follows immediately that

$$A_\theta=2\pi x/(\sqrt{A_2/A_1}\cos^2\theta+\sqrt{A_1/A_2}\sin^2\theta). \tag{8}$$

Equations (7) and (8) were derived for the Gaussian model. However, their validity extends to all critical models in the Gaussian universality class, which includes almost all known two-dimensional models (see e.g. Kadanoff and Brown 1979, Knops 1980 and den Nijs 1981 and references therein). This is shown by the following argument.

Finite-size scaling can be derived from renormalisation group theory, assuming that $1/n$ is an additional scaling field (Suzuki 1977, Blöte and Nightingale 1982). To be precise, the assumption is that, if corrections to scaling are ignored, the renormalisation group equations of the finite system are the same as those of the system in the thermodynamic limit; the scaling of $1/n$ is an immediate consequence of the length rescaling under renormalisation. Equation (1) follows if the only non-zero, relevant scaling field is $1/n$. Also, the universality of the amplitude A is obtained from this assumption; i.e. the presence of non-zero irrelevant scaling fields shows up only in corrections to scaling of the $1/n$ behaviour of the inverse correlation length. Finally, the argument is completed by observing that models within the same universality class differ only by the values of the irrelevant scaling fields.

Note that A in (5) is a universal amplitude, while in general only *ratios* of amplitudes are universal. This is a consequence of the fact that physical parameters couple to scaling fields in a system-dependent way, and this is where $1/n$ is an exception. Generally, unknown constants are introduced into amplitudes, and these cancel only in suitably chosen amplitude ratios.

The discussion above holds for correlations of operators which can be identified with spin-wave operators in the Gaussian model. It also applies to the case of the dual vortex operators, but not necessarily to the mixed spin-wave–vortex case. We shall treat various different operators below; indices will be used to identify the associated amplitudes and anomalous dimensions.

From the assumption that, with respect to critical behaviour, lattice and continuum models differ only in an irrelevant way (in the renormalisation group sense) it follows that equation (8) for the general strip orientation is valid also for lattice models. Furthermore, we expect its validity to extend to general lattices such as honeycomb or triangular. The angle θ is then defined with respect to one of the principal directions. These generically depend in an unknown way on the anisotropic interactions.

Spin–spin correlations in the Ising model canot be related in any obvious way to correlations in the Gaussian model that are of simple spin-wave or vortex type. Therefore our derivation does not apply directly to this case. Yet (7) does hold for this model, as we now show. Again the two interaction constants in the square lattice are denoted by K_1 and K_2. A straightforward calculation, starting from the exact solution of Onsager (see e.g. Domb 1960), gives

$$A_{1\mathrm{m}} = \tfrac{1}{4}\pi(\sinh 2K_2/\sinh 2K_1)^{1/2} \tag{9}$$

for the amplitude of the inverse correlation length in the K_1 direction. At criticality one has $(\exp 2K_1 - 1)(\exp 2K_2 - 1) = 2$. Interchanging subindices 1 and 2, we find that $A_{1\mathrm{m}}$ and $A_{2\mathrm{m}}$ indeed satisfy (7), since x_m, the anomalous dimension of the order parameter, equals $\frac{1}{8}$.

Equation (9) was obtained for the correlation length in the Ising model associated with spin–spin correlations. The largest and second-largest eigenvalues of the transfer matrix are the ones that appear in the expression for this length. They are found in the sectors which are respectively even and odd under spin inversion. In the case of energy–energy correlations the pertinent eigenvalues are both in the even sector. Again from the exact solution one finds

$$A_{1\mathrm{T}} = 2\pi(\sinh 2K_2/\sinh 2K_1)^{1/2}, \tag{10}$$

in agreement with (7) and $x_\mathrm{T} = 1$ for the anomalous dimension of the energy. The independence of anisotropy of the ratio $A_{1\mathrm{m}}/A_{1\mathrm{T}}$ is likely to be a universal feature.

Next we present the results of numerical calculations. First we consider the symmetric eight-vertex, or Baxter, model (Baxter 1972). Formulated in terms of Ising spins $s_i = \pm 1$, the reduced Hamiltonian of this model reads

$$\mathcal{H} = K_2 \sum_{(i,j)} s_i s_j + K_4 \sum_{(i,j,k,l)} s_i s_j s_k s_l, \tag{11}$$

the first sum being over next-nearest-neighbour bonds on a square lattice, the second over elementary plaquettes. There are three relevant exponents: the thermal anomalous dimension x_T; and x_m and x_p, which pertain to magnetisation and polarisation. They are given by Baxter (1972), Barber and Baxter (1973), Baxter and Kelland (1974) and Baxter (1980):

$$x_T = (2/\pi)\cos^{-1}\tanh 2K_4, \qquad x_p = \tfrac{1}{4}x_T, \qquad x_m = \tfrac{1}{8}. \qquad (12a, b, c)$$

We found the associated amplitudes A_T, A_m and A_p of the inverse correlation lengths from the ratios of the largest eigenvalue of the transfer matrix and the largest subdominant eigenvalues belonging to eigenvectors which have the following symmetry properties. In the magnetic case the desired eigenvector is odd under spin inversion. Similarly, A_p is obtained by imposing oddness under polarisation inversion, i.e. by flipping a sublattice of next-nearest-neighbour spins. The eigenvector associated with the thermal amplitude A_T is even under both of these transformations.

Figures 1(a), (b) and (c) are plots of $n\kappa_n - A$ against x_T for $n = 4, 6, \ldots, 16$, where A is obtained from the relations (7) and (12). The poor convergence at both extremes of x is to be expected in view of previous calculations (Nightingale 1977). In table 1 extrapolated estimates of the amplitudes are compared with the conjectured exact values. Assuming power law convergence according to $A = n\kappa_n + cn^b$, estimates of A can be obtained from three consecutive values of κ_n (Blöte and Nightingale 1982). Such three-point fits were made to the values obtained for $n = 12$, 14 and 16. The results clearly indicate that in the Baxter model equation (7) is satisfied for the thermal, magnetic and polarisation amplitudes.

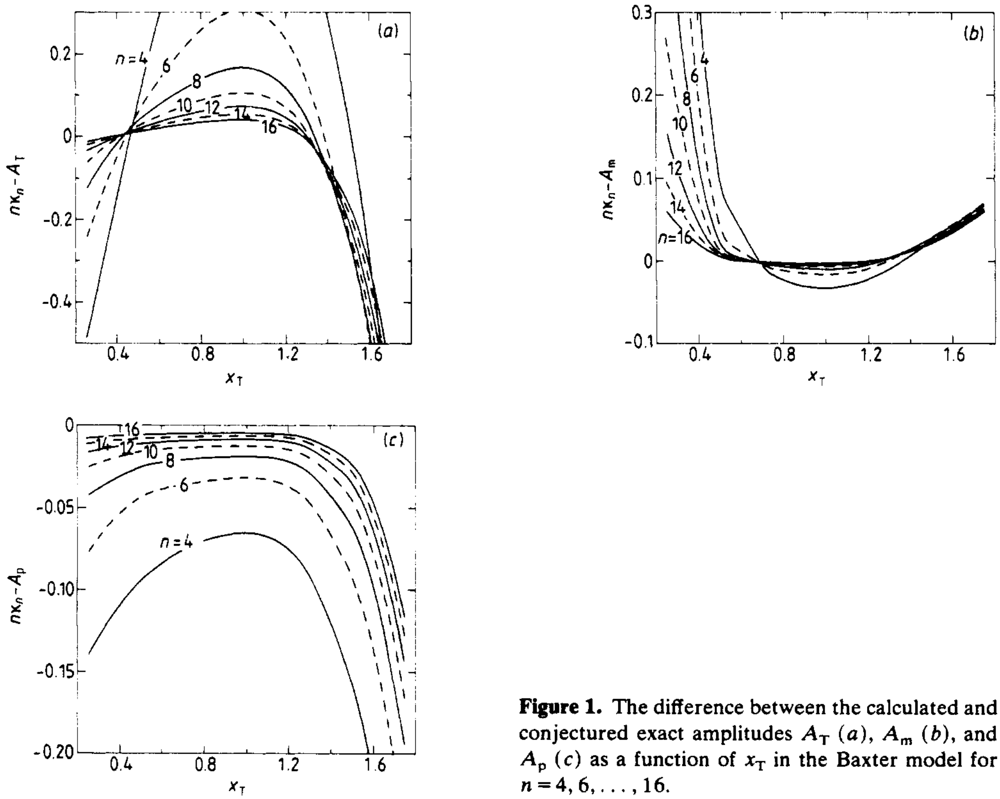

Figure 1. The difference between the calculated and conjectured exact amplitudes A_T (a), A_m (b), and A_p (c) as a function of x_T in the Baxter model for $n = 4, 6, \ldots, 16$.

Table 1. Three-point ($n=12$, 14, 16) extrapolated estimates (where possible) of the temperature, magnetic and polarisation amplitudes are compared with the conjectured exact values, obtained on the basis of Baxter's exact results, for various values of the four-spin interaction of the Baxter model.

K_4	A_T	Exact	A_m	Exact	A_p	Exact
−0.8					2.7249	2.7432
−0.4					2.2972	2.2970
−0.2	7.8409	7.8422	0.7867	0.7854	1.9607	1.9605
−0.1	7.0788	7.0779	0.7855	0.7854	1.7696	1.7695
0.0	6.2839	6.2832	0.7855	0.7854	1.5710	1.5708
0.1	5.4892	5.4885	0.7855	0.7854	1.3723	1.3721
0.2	4.7248	4.7242	0.7854	0.7854	1.1812	1.1811
0.4	3.3781	3.3784	0.7856	0.7854	0.8447	0.8446
0.8	1.5919	1.5937	0.7857	0.7854	0.3975	0.3984

The next model for which we present numerical results is the continuous q-state Potts model, again on the square lattice. We used the formulation of the model as a Whitney polynomial (Kasteleyn and Fortuin 1969, Baxter 1973). The correlation lengths were computed by employing a transfer matrix as introduced in previous finite-size calculations for the Potts model (Blöte *et al* 1981, Blöte and Nightingale 1982). For the thermal amplitude A_T we made use of the transfer matrix of the *simple* Whitney polynomial (i.e. no ghost site included). The case of the magnetic amplitude A_m was treated with the *extended* polynomial (i.e. including a ghost site).

The thermal and magnetic exponents of the Potts model are given by the generally accepted conjectures (den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980, den Nijs 1981, Black and Emery 1981, Nienhuis 1982a, den Nijs 1983)

$$x_T = 3/x - 1, \qquad x_m = (1-y^2)/4x \tag{13a, b}$$

with $x = 2 - y$ and $\cos(\pi y/2) = \frac{1}{2}\sqrt{q}$.

In figures 2(a, b) we show the differences $n\kappa_n - A$ of the estimated and exact values of the amplitudes against x_T. This was done for $n = 2, 3, \ldots, 10$ and $n = 2, 3, \ldots, 8$

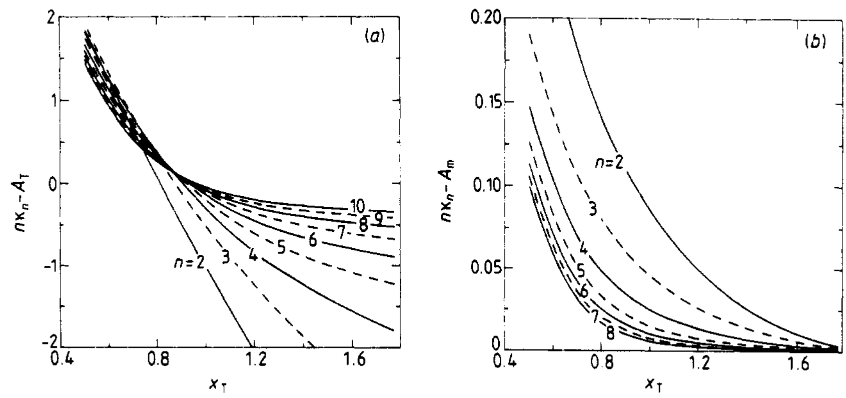

Figure 2. The difference between the calculated and conjectured exact amplitudes A_T (a), and A_m (b) as a function of x_T in the Potts model for $n = 2, 3, \ldots, 10$ (thermal) and $n = 2, 3, \ldots, 8$ (magnetic).

in the thermal and magnetic cases respectively. Results for A_T of three-point extrapolations are shown in table 2. For the magnetic case, we also checked relation (7) for anisotropic lattices. Table 3 contains the extrapolated results. As in the Baxter model, the numerical results agree with equation (7).

Table 2. Three-point ($n = 8, 9, 10$) extrapolated estimates of the temperature amplitude A_T are compared with the conjectured exact results for various values of the number of states q of the Potts model.

q	A_T	Exact
0.0625	11.193	11.174
0.95	7.973	7.953
1.05	7.775	7.758
2.00	6.284	6.283
4.00	3.240	3.142

Table 3. Three-point ($n = 6, 7, 8$) extrapolated estimates of the magnetic amplitude A_m. For different values of the anisotropy $(e^{K_2}-1)/(e^{K_1}-1)$, the geometric mean A_m of A_{1m} and A_{2m} is compared with the conjectured exact results for various values of the number of states q of the Potts model.

Anisotropy	$\sqrt{2}$		2		4		1	
q	A_{1m}	A_m	A_{1m}	A_m	A_{1m}	A_m	A_m	Exact
0.0625	0.263 19	0.222 90	0.310 78	0.222 89	0.433 21	0.222 75	0.222 90	0.222 87
0.25			0.551 55	0.400 39			0.400 41	0.400 32
0.95			0.871 27	0.644 43			0.644 49	0.644 32
1.05			0.896 57	0.664 29			0.664 35	0.664 18
2.00	0.907 52	0.785 94	1.048 17	0.785 89	1.398 79	0.784 88	0.785 94	0.785 40
3.00			1.115 92	0.842 82			0.842 77	0.837 76
4.00			1.132 21	0.857 24			0.856 97	0.785 40

Finally we also applied analogous numerical techniques to the N-component cubic model on an $n\times\infty$ strip. The reduced Hamiltonian is a sum over all pairs of nearest-neighbour sites of a square lattice

$$\mathscr{H} = K \sum_{(i,j)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \qquad (14)$$

where the $\boldsymbol{\sigma}$ are discrete vectors of unit length and $2N$ possible orientations: parallel or antiparallel to N Cartesian axes. We (Blöte and Nightingale, to be published) constructed a transfer matrix for this model which treats N as a continuous parameter. A thermal correlation length (analogous to the one of the simple Whitney model mentioned above) was calculated for linear system sizes up to $n = 8$, and for $N = \frac{1}{64}, \frac{1}{32}, \ldots, \frac{1}{2}$ ($N = 1$ and $N = 2$ reduce to Ising models). Critical couplings were obtained by scaling of the correlation length (see e.g. Nightingale 1982). The results showed good apparent convergence; estimates are shown in table 4. These data were used to compute the amplitude A_T. Again, the apparent convergence is good, and the results together with conjectured exact values are listed in table 4. The exact numbers were

Table 4. Finite-size results for the critical coupling strength and for the amplitude of the N-component cubic model. Differences from the conjectured exact amplitude are of the same order as numerical uncertainties due to extrapolation.

N	K_c	A_T	Exact
$\frac{1}{64}$	0.005 935	4.241	4.217
$\frac{1}{32}$	0.011 89	4.269	4.245
$\frac{1}{16}$	0.023 91	4.325	4.301
$\frac{1}{8}$	0.048 29	4.439	4.414
$\frac{1}{4}$	0.098 46	4.672	4.647
$\frac{1}{2}$	0.204 6	5.167	5.138

obtained from (7) and a conjecture of Cardy and Hamber (1980, see also Nienhuis 1982b), according to which the thermal anomalous dimension reads ($-2<N<2$)

$$x_T = 4\pi/[2\pi - \cos^{-1}(-N/2)] - 2. \tag{15}$$

The agreement between estimated and conjectured results is close. Assuming, in view of the evidence presented above for other models, that (7) holds for the cubic model, this implies a verification of the exponent conjecture. We also mention that independent calculation of x_T from temperature derivatives of free energy and correlation length agrees with the above results to within a few parts in one hundred, consistent with apparent uncertainties in the extrapolation procedures.

In conclusion, all our results, both analytic and numerical, support equation (7), which establishes a relation between finite-size amplitudes of inverse correlation lengths associated with correlations of various types of operators and the corresponding anomalous dimensions.

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